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Large Orbits of Supersolvable Linear Groups

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The study of regular orbits of linear groups plays an important role in representation theory, particularly that of solvable groups because a chief factor of a solvable group G is an irreducible G-module. If V is a G-module, recall that v in V is in a regular orbit if $\mathbf{C}_G(v) = 1$, i.e., the G-orbit of v is as large as possible and it has size $|G|$. All groups considered here are finite. Furthermore, we only consider finite vector spaces, since there is otherwise always a regular orbit and because this is where the interesting applications lie. Existence of regular orbits has had applications to Brauer's conjectures on height zero characters and block size as well as length-type problems. Even if G is nilpotent, G need not have a regular orbit (e.g., see Examples 4.5 in [MW]). Passman [Pa] shows that if G is a p-group and V completely reducible, then there exists an orbit as large as $|G|^{1/2}$ by proving $\mathbf{C}_G(x) \cap \mathbf{C}_G(y) = 1$ for some x and y in V. He also uses this to prove results about Sylow intersections in solvable groups. "Added in proof" at the end of Passman's paper is a remark that the techniques produce an even larger orbit. Proofs of this have been given by Passman in correspondence and by Isaacs [Is] with a slight strengthening and different technique. Specifically, Isaacs shows that if G is nilpotent and V is a completely reducible faithful G-module, then there is v in V such that $|\mathbf{C}_G(v)| \le (|G|/p)^{1/p}$ for the smallest prime divisor p of $|G|$. For G nilpotent, complete reducibility is equivalent to $(|G|, |V|) = 1$. In his paper, Isaacs asks whether there is always an orbit as large as $|G|^{1/2}$ whenever G is solvable and $(|G|, |V|) = 1$ and he also posed this question in correspondence for G supersolvable. We answer the latter affirmatively without any coprimeness hypothesis. Our main result is:

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THEOREM A. *If V is a faithful completely reducible G-module for a supersolvable group G, then there exist x and y in V such that $\mathbf{C}_G(x) \cap \mathbf{C}_G(y) = 1$.*

An immediate corollary is that there is G-orbit in V with size at least $|G|^{1/2}$. We do note that Theorem A does not remain true if supersolvable is replaced by solvable. This is because there exist irreducible G-modules V with G solvable where $|G| > |V|^2$ (see [MW, Example 3.8]). If G is solvable and $(|G|, |V|) = 1$, then $|G| < |V|^2$ (see [MW, Theorem 3.5(b); Pf, Theorem 1]) and so it is possible that Theorem A extends to this case. Robinson [Ro] has shown that Brauer's block size conjecture for p-solvable G (also known as the $k(GV)$-conjecture) can be proven if under appropriate conditions $\mathbf{C}_G(v)$ has a regular module on V for some $v \in V$. Our conclusion is a little weaker, namely there is an $x \in V$ such that $\mathbf{C}_G(x)$ has a regular orbit in V.

The supersolvable primitive linear groups are semi-linear groups (see Lemma 7) and these groups have no regular orbits on their respective modules. In particular, these groups and linear groups induced from them pose an obstacle to finding large orbits (e.g., see [Tu]). Much of this paper goes towards understanding what happens to linear groups induced from semi-linear groups. But we will use supersolvability even more, as we will give an example of a case where Theorem A fails for a group G that is the wreath product of two supersolvable groups. Our first proposition is quite trivial, but quite essential to our work. Even when $w = 0$, it has meaning and use.

PROPOSITION 1. *Suppose that V is a faithful G-module and that $V = W_1 \oplus \cdots \oplus W_n$ for subspaces W_i that are permuted by G. Suppose that $w_i, x_i, y_i \in W_i$ and $w = w_1 + \cdots + w_n$. If x_k and y_k are $\mathbf{C}_G(w)$-conjugate, then x_k and y_k are conjugate in $\mathbf{C}_H(w_k)$ where $H = \mathbf{N}_G(W_k)$. In particular, x_k can only be $\mathbf{C}_G(w)$-conjugate to elements of one $\mathbf{C}_H(w_k)/\mathbf{C}_G(W_k)$-conjugacy class in W_k.*

Proof. We may assume that $x_k \neq 0$. If $g \in \mathbf{C}_G(w)$ and $x_k^g = y_k$, then $W_k^g = W_k$ and even $g \in \mathbf{C}_G(w_k) \cap H = \mathbf{C}_H(w_k)$. Of course, $\mathbf{C}_H(w_k) = \mathbf{C}_G(w_k)$ if $w_k \neq 0$. The second statement follows from the first. ∎

LEMMA 2. *Suppose that V is a faithful G-module and that $V_C = W_1 \oplus \cdots \oplus W_n$ for C-invariant subspaces W_i that are permuted by G/C. Assume that G/C is cyclic and has a faithful orbit on $\{W_1, \ldots, W_n\}$. Suppose that $w_i \in W_i$ and that $\mathbf{C}_G(w_i)/\mathbf{C}_G(W_i)$ has at least $r > 1$ regular orbits on W_i for each i. Set $w = w_1 + \cdots + w_n$ and let $k = |C\mathbf{C}_G(w)/C|$. Then $\mathbf{C}_G(w)$ has at least r regular orbits on V except possibly when $r = n = k = 2$ and $\mathbf{C}_G(w)$ has exactly one regular orbit on V. Furthermore $\mathbf{C}_G(w)$ has at least 3 regular orbits on V except possibly when $r = 2$ and $(k, n) \in \{(1,1), (2,2), (2,3), (3,3)\}$.*

Proof. If $r > 2$, then we may choose s_i, t_i, and u_i in distinct regular orbits of $\mathbf{C}_G(w_i)/\mathbf{C}_G(W_i)$. If $r = 2$, we just choose s_i and t_i. If w_i and w_k are $\mathbf{C}_G(w)$-conjugate, then we may assume that s_i and s_k are also $\mathbf{C}_G(w)$-conjugate as well as t_i and t_k and also u_i and u_k. By Proposition 1, it follows that s_i is $\mathbf{C}_G(w)$-conjugate to no t_k or u_k; and that no t_i is $\mathbf{C}_G(w)$-conjugate to a u_k. If $x_i, y_i \in \{s_i, t_i, u_i\}$, then x and y are in regular orbits of $\mathbf{C}_C(w)$ where $x = x_1 + \cdots + x_n$ and $y = y_1 + \cdots + y_n$, because each W_i is C-invariant and V is a faithful C-module. Furthermore, w and y are not conjugate under $\mathbf{C}_C(w)$ unless $x_i = y_i$ for all i. So $\mathbf{C}_C(w)$ has at least r^n regular orbits on V. If $\mathbf{C}_G(w) \subseteq C$, we are done. Letting $L/C = C\mathbf{C}_G(w)/C \cong \mathbf{C}_G(w)/\mathbf{C}_C(w)$, we may assume that $L > C$ and $k > 1$.

We may assume that $\{W_1, \ldots, W_k\}$ is a regular orbit of L/C and so $C = \mathbf{N}_L(W_1)$. Let $z = s_1 + x_2 + \cdots + x_n$ with x_i in a regular orbit of $\mathbf{C}_G(w_i)/\mathbf{C}_G(W_i)$, but such that x_i is not $\mathbf{C}_G(w_i)$-conjugate to s_i for $2 \le i \le k$. From Proposition 1, it follows that $\mathbf{C}_G(w) \cap \mathbf{C}_G(z) \subseteq \mathbf{N}_L(W_1) = C$ and thus $\mathbf{C}_G(w) \cap \mathbf{C}_G(z) = \mathbf{C}_C(w) \cap \mathbf{C}_C(z) = 1$, with the last equality from the last paragraph. So z is in a regular orbit of $\mathbf{C}_G(w)$. Likewise, so is $t_1 + x_2 + \cdots + x_n$ provided x_i is in a regular orbit of $\mathbf{C}_G(w_i)/\mathbf{C}_G(W_i)$ and x_i is not $\mathbf{C}_G(w_i)$-conjugate to t_i for $2 \le i \le k$. Thus there are at least $r^{n-k+1}(r-1)^{k-1}$ distinct elements lying in regular orbits of $\mathbf{C}_G(w)$ which lie in distinct orbits of $\mathbf{C}_C(w)$. Because $|\mathbf{C}_G(w)/\mathbf{C}_C(w)| = k$, there are at least $r^{n-k+1}(r-1)^{k-1}/k$ distinct regular orbits of $\mathbf{C}_G(w)$ on V. For $r > 2$, then $\mathbf{C}_G(w)$ has at least $r2^{k-1}/k \ge r$ regular orbits on V. We now assume that $r = 2$. If $k = 2$, we have at least $2^{n-1}/2$ regular orbits of $\mathbf{C}_G(w)$ on V and the conclusion of this lemma is satisfied. For $k > 2$, the element $t_1 + s_2 + t_3 + \cdots + t_k + x_{k+1} + \cdots + x_n$ has not been included in the above count of elements in regular orbits of $\mathbf{C}_G(w)$ and we even get $k2^{n-k+1}/k \ge 2^{n-k+1}$ regular orbits of $\mathbf{C}_G(w)$ on V. The conclusion of this lemma is satisfied except when possibly when $n = k > 3$. We have shown the existence of two regular orbits of $\mathbf{C}_G(w)$ on V. It suffices to produce a third regular orbit of $\mathbf{C}_G(w)$ on V and there is one of the form $s_1 + s_2 + \cdots + s_m + t_{m+1} + \cdots + t_n$ if $\mathbf{N}_G(W_1 + \cdots + W_m) = C$ and $1 < m < n - 1$. Hence, for the regular transitive action of the cyclic group L/C on $\{1, \ldots, n\}$, it suffices to show the existence of some $\Delta \subseteq \{1, \ldots, n\}$ with $1 < |\Delta| < n - 1$ such that $\mathrm{Stab}_{L/C}(\Delta) = 1$. Each $\Delta \subseteq \{1, \ldots, n\}$ is the union of faithful orbits of $\mathrm{Stab}_{L/C}(\Delta)$ and so $|\mathrm{Stab}_{L/C}(\Delta)|$ divides $|\Delta|$ and $|L|$. For $n \notin \{4, 6\}$, $\varphi(n) > 2$ for Euler's φ-function and so there is some Δ in with $\mathrm{Stab}_{L/C}(\Delta) = 1$ and $1 < |\Delta| < n - 1$. Such a set Δ can also be found when $n = 4$ because only two of the six subsets of $\{1, 2, 3, 4\}$ that have cardinality two do have non-trivial stabilizers in L/C. When $n = 6$, such a set Δ can also be found because only three of the fifteen subsets of $\{1, 2, 3, 4, 5, 6\}$ that have cardinality two do have non-trivial stabilizers in L/C. ∎

A solvable primitive permutation group S on Ω has a unique minimal normal subgroup M. Furthermore, M transitively and regularly permutes the elements of Ω so that $|\Omega| = |M|$ is a prime power. For each α in Ω, we have that $S = MS_\alpha$ with $M \cap S_\alpha = 1$ and that the actions of S_α on M and Ω are permutation isomorphic. A group is supersolvable if every chief factor has prime order. If S is supersolvable, then $|M| = |\Omega| = p$ for a prime p and S is isomorphic to a subgroup of the semi-direct product $Z_p \circ Z_{p-1}$ that contains Z_p. In particular, $S_\alpha \cap S_\beta = 1$ for distinct α, $\beta \in \Omega$.

COROLLARY 3. *Suppose that V is a faithful G-module and that $V_C = W_1 \oplus W_2 \oplus \cdots \oplus W_p$ for C-invariant submodules W_i that are permuted faithfully and primitively by the supersolvable factor group G/C. Suppose that $w_i \in W_i$ and set $w = w_1 + w_2 + \cdots + w_p$. If $\mathbf{C}_G(w_i)/\mathbf{C}_G(W_i)$ has at least $r \geq 3$ regular orbits on W_i for each i, then $\mathbf{C}_G(w)$ has a regular orbit on V. Indeed, then $\mathbf{C}_G(w)$ has at least r regular orbits on V except possibly when $r = 3 = p$ and $G/C \cong S_3$.*

Proof. By Lemma 2, we may assume that $C\mathbf{C}_G(w)/C$ is not cyclic and thus $p > 2$. Since G/C is isomorphic to a subgroup of $Z_p \circ Z_{p-1}$, it follows that $C\mathbf{C}_G(w)/C$ is transitive on the W_i. We may choose s_i, t_i, and u_i in distinct regular orbits of $\mathbf{C}_G(w_i)/\mathbf{C}_G(W_i)$ and assume by Proposition 1 that no two of s_i, t_j, or u_k are ever $\mathbf{C}_G(w)$-conjugate. Now $\mathbf{C}_G(w) \cap \mathbf{C}_G(s)$ for $s = s_1 + t_2 + u_3 + \cdots + u_p$ must stabilize both W_1 and W_2 and thus lie in C (see comments preceding this Corollary). Thus $\mathbf{C}_G(w) \cap \mathbf{C}_G(s) \subseteq \mathbf{C}_C(w) \cap \mathbf{C}_C(s) \subseteq \cap_i \mathbf{C}_C(W_i) = 1$. We have that $s = s_1 + t_2 + u_3 + \cdots + u_p$ lies in a regular orbit of $\mathbf{C}_G(w)$, as desired. In fact, this argument shows there exist $\binom{r}{3} = r(r-1)(r-2)/6$ elements of V which lie in distinct regular orbits of $\mathbf{C}_G(w)$. For $r > 3$, $\mathbf{C}_G(w)$ has at least r regular orbits on V. To complete the proof, we need just show that $\mathbf{C}_G(w)$ has at least 3 regular orbits on V when $r = 3$ and $3 < p$. But then we have that s, $s_1 + u_2 + t_3 + \cdots + t_p$ and $t_1 + u_2 + s_3 + \cdots + s_p$ lie in three distinct regular orbits of $\mathbf{C}_G(w)$. ∎

Gluck's Permutation Lemma [MW, Theorem 5.6] characterizes those solvable primitive permutation groups (S, Ω) for which S does not have a regular orbit on the power set of Ω. In all these cases, $|\Omega| < 10$. An important case in the proof is where $|\Omega|$ is prime, i.e., S is supersolvable. We quote this.

LEMMA 4 (Gluck). *If S is a supersolvable primitive permutation group on Ω, then there is a subset $\Delta \subseteq \Omega$ such that* $\mathrm{Stab}_S(\Delta) = 1$ *unless* $|\Omega| =$ 3, 5, *or* 7.

DEFINITION. If V is a G-module, we say that $v \in V$ lies in a semi-regular G-orbit if $\mathbf{C}_G(v)$ has a regular orbit on V. Furthermore, we say that v is in a type k semi-regular orbit if $\mathbf{C}_G(v)$ has at least k regular orbits on V.

LEMMA 5. *Suppose that V is a faithful G-module and that $V_C = W_1 \oplus W_2 \oplus \cdots \oplus W_p$ for C-invariant submodules W_i that are permuted faithfully and primitively by the supersolvable factor group G/C. Suppose that $w_1, x_1 \in W_1$ lie in distinct $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$-semi-regular orbits of types r and s. If $rs > 2$ or if $rs = 2 = p$, then there exist u and v in distinct semi-regular G-orbits of V of types ρ and σ with $\rho\sigma \geq rs$.*

Proof. Choose $w_i, x_i \in W_i$ that are G-conjugate to w_1 and x_1, respectively. Now each w_i (respectively, x_i) is a type r (resp. type s) semi-regular orbit of $\mathbf{N}_G(W_i)/\mathbf{C}_G(W_i)$. Also no w_i is G-conjugate to an x_j by Proposition 1 applied to $\mathbf{C}_G(0)$. We may assume that $G/C \neq 1$ and thus that p is prime.

Assume first that $p \notin \{3, 5, 7\}$ so that we may apply Gluck's lemma and choose (without loss of generality) m with $1 \leq m < p$ such that $\mathrm{Stab}_{G/C}\{1, \dots, m\} = 1$. If $u = w_1 + \cdots w_m + x_{m+1} + \cdots + x_p$, then $\mathbf{C}_G(u) \subseteq C$ and $\mathbf{C}_G(u) = \mathbf{C}_C(u)$ has at least $r^m s^{p-m} \geq rs$ regular orbits on V. Also $\mathbf{C}_G(v)$ has at least $s^m r^{p-m} \geq rs$ regular orbits on V where $v = x_1 + \cdots x_m + w_{m-1} + \cdots + w_p$. Furthermore u and v are not conjugate in G if p is odd. If $p = 2$, then $w = w_1 + w_2$ is in a semi-regular orbit of G by Lemma 2 and w is not conjugate to u by Proposition 1. Thus, we may now assume that $p = 3$, 5, or 7. By the hypotheses, $rs > 2$. We may assume that $r \geq s$, and so $r > 2$ or $r = 2 = s$.

Now let $u^* = w_1 + \cdots + w_{p-1} + x_p$ and $v^* = w_1 + \cdots + w_{p-2} + x_{p-1} + x_p$, so that u^* and v^* are not G-conjugate. Now $\mathbf{C}_G(u^*)C/C$ and $\mathbf{C}_G(v^*)C/C$ are both cyclic. If $r = s = 2$, then Lemma 2 shows that $\mathbf{C}_G(u^*)$ and $\mathbf{C}_G(v^*)$ each have (at least) two regular orbits on V. We thus assume that $r > 2$. Observe that $\mathbf{C}_G(u^*)$ must stabilize W_p and $C\mathbf{C}_G(u^*)/C$ faithfully permutes $\{W_1, \dots, W_{p-1}\}$. Lemma 2 shows that $\mathbf{C}_G(u^*)/\mathbf{C}_G(u^*) \cap \mathbf{C}_G(W_1 + \cdots + W_{p-1})$ has at least r regular orbits on $W_1 + \cdots + W_{p-1}$. Since $\mathbf{C}_G(u^*) \subseteq \mathbf{N}_G(W_p)$, we have that $\mathbf{C}_G(u^*)/\mathbf{C}_G(u^*) \cap \mathbf{C}_G(W_p)$ has at least s regular orbits on W_p. Hence $\mathbf{C}_G(u^*)$ has at least rs regular orbits on $V = W_1 + \cdots + W_p$. Also Corollary 3 shows that $\mathbf{C}_G(w)$ has a regular orbit on V where $w = w_1 + w_2 + \cdots + w_p$. Because u^* and w are not conjugate, the proof is complete. ∎

COROLLARY 6. *Suppose that V is a faithful G-module and that $V_C = W_1 \oplus W_2 \oplus \cdots \oplus W_p$ for C-invariant submodules W_i that are permuted faithfully and primitively by the supersolvable factor group G/C. Assume that $C \neq 1$ and that $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$ has a regular orbit on W_1. Then there exist u and v in distinct semi-regular G-orbits of V of types ρ and σ with $\rho\sigma \geq |W_1| > 2$.*

Proof. Since $C \neq 1$, indeed $|W_1| > 2$. If $w \in W_1$ is in a regular orbit $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$, then $w \neq 0$ lies in a type $|W_1|$ semi-regular of $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$, while 0 does lie in a different semi-regular orbit of $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$. Apply Lemma 5. ∎

Suppose that V is a vector space of dimension n over $GF(q)$. We may then label the elements of V by those of $GF(q^n)$ in a one-to-one fashion. We let $\Gamma(V)$ or $\Gamma(q^n)$ denote the group of semi-linear transformations of V, i.e., $\Gamma(V) = \{x \to ax^\sigma | 0 \neq a \in GF(q^n),\ \sigma \in \mathrm{Gal}(GF(q^n)/GF(q))\}$. In particular, $\Gamma(V)$ has a cyclic normal subgroup $\Gamma_0(V)$ of order $q^n - 1$ that acts fixed-point-freely on V (by multiplications) and the factor group $\Gamma(V)/\Gamma_0(V)$ is cyclic of order n (isomorphic to the Galois group). In particular, $\Gamma(V)$ is metacyclic. For a faithful G-module V, we will say $G \leq \Gamma(V)$ if the elements of V may be labeled in a way that yields G a subgroup of $\Gamma(V)$. The semi-linear groups $\Gamma(V)$ are supersolvable and play an important role in representation theory. We gather some known facts about supersolvable groups, the last of which characterizes primitive supersolvable linear groups as semi-linear groups.

LEMMA 7. *Let G be supersolvable. Then*

(a) *$G/\mathbf{F}(G)$ is abelian;*

(b) *G has a normal Sylow-p-subgroup if p is the largest prime divisor of $|G|$; and*

(c) *If V is a faithful primitive and finite G-module, then $G \leq \Gamma(V)$.*

Proof. Parts (a) and (b) are immediate from the definition of supersolvability and are well known. Let A be a maximal abelian normal subgroup of G. Again, it is well known and direct from the definition of supersolvable that $A = \mathbf{C}_G(A)$. Since V is a primitive G-module and A is normal in G, V_A is homogeneous by Clifford's Theorem. Because V_A is homogeneous and $A = \mathbf{C}_G(A)$, it follows that $G \leq \Gamma(V)$ (e.g., see [MW, Lemma 2.2]). ∎

Recall, for positive integers $q > 1$ and n, a prime divisor r of $q^n - 1$ is called a *Zsigmondy prime divisor* of $q^n - 1$ if r does not divide $q^m - 1$ whenever $0 < m < n$. In this case, n is the order of $q \pmod r$ and thus n divides $\phi(r) = r - 1$ for Euler's ϕ-function. In particular, $\gcd(r, n) = 1$. The Zsigmondy Prime Theorem asserts that $q^n - 1$ always has a Zsigmondy prime divisor except when $q^n = 2^6$ or when $n = 2$ and q is a Mersenne prime.

LEMMA 8. *Suppose that V is a faithful FG-module with G supersolvable and that G acts transitively on $V - \{0\}$. Set $|V| = q^n$ where $q = |F|$.*

(i) *Then $G \subseteq \Gamma(V)$; and*

(ii) *If r is a Zsigmondy prime divisor of $q^n - 1$ and $R \in \mathrm{Syl}_r(G)$, then* $R \subseteq \Gamma_0(V) \cap G = \mathbf{F}(G) = \mathbf{C}_G(R)$;

(iii) *If $q^n = 2^6$, then $\mathbf{F}(G) = G \cap \Gamma_0(V)$ is cyclic of order 21 or 63.*

Proof. The transitivity hypothesis implies that V has no proper G-modules and is thus irreducible. Suppose that $V = V_1 \oplus \cdots \oplus V_n$ for modules V_i that are permuted by G. If $v_i \in V_i$ is non-zero for each i, then v_1 cannot be G-conjugate to $v_1 + v_2$. Thus the transitivity hypothesis implies that $n = 1$ and that V is a primitive G-module. By Lemma 7(c), $G \subseteq \Gamma(V)$. If r is a Zsigmondy prime divisor of $q^n - 1$ and $R \in \mathrm{Syl}_r(G)$, then $1 \neq R$ by transitivity and also $R \subseteq \Gamma_0(V)$ because $\gcd(r, n) = 1$ (see paragraph preceding this lemma). Statements of (ii) and (iii) are immediate from Lemmas 6.4 and 6.5(b), respectively, of [MW]. ∎

PROPOSITION 9. *If $G = \Gamma(V)$ and $0 \neq x \in V$, then $\mathbf{C}_G(x)$ has at least three non-zero regular orbits on V unless $|V| = 2^n$ with $n \leq 3$ or $|V| = 3$.*

Proof. Set $|V| = q^n$ for a prime power q. If $n = 1$, then $\mathbf{C}_G(x) = 1$ and the result is trivial. Now G acts transitively on $V^{\#}$ and so we may replace x by any non-zero element of V. If τ is a field automorphism of order n, then $\langle \tau \rangle$ is the centralizer of an element of V and so we may assume that $\mathbf{C}_G(x) = \langle \tau \rangle$. Now $C = \mathbf{C}(x)$ is cyclic of order n. If n is prime, then $\mathbf{C}_V(C)$ has order q and every element in $V - \mathbf{C}_V(C)$ is in a regular orbit of C. Since $(q^n - q)/n \geq 3$ unless $q = 2$ and $n \leq 3$, the result follows when n is prime.

If n is a power of two, then C has a unique involution t. Every element of $V - \mathbf{C}_V(t)$ is in a regular orbit of C. Since $|\mathbf{C}_V(t)| = q^{n/2}$ and $n > 3$, there exist at least $(q^n - q^{n/2})/n \geq 2^{n/2}(2^{n/2} - 1)/n \geq 3$ regular orbits of C. The result holds for n a power of two.

If w is a generator of the cyclic multiplicative group $GF(q^n)^{\#}$, then w is fixed by no non-trivial field automorphism and hence lies in a regular orbit of C. Thus every element of the set U of generators of $GF(q^n)^{\#}$ is in a regular orbit of C and n divides $|U| = \varphi(q^n - 1)$, where φ is Euler's phi function. The proof is completed once we establish that $2n < \varphi(q^n - 1)$ for composite $n > 5$.

We may assume that $q^n \neq 2^6$ and n is composite, so that we may we choose a Zsigmondy prime divisor p of $q^n - 1$. Observe that $o(q) = n$ (mod p) and so n divides $p - 1$, which in turn must divide $\varphi(q^n - 1)$. So we may assume that $\varphi(q^n - 1)$ is $p - 1$ or $2(p - 1)$. Factoring $q^n - 1$ as a product of primes (including p), applying the multiplicativity of the phi-function, and noting that $p > n > 5$, this can only happen if $q^n - 1$ divides $4p$ or $6p$. Write $n = st$ with $s, t > 1$. Since p is a Zsigmondy prime divisor of $q^n - 1$, it follows that $q^s - 1$ and $q^t - 1$ are proper divisors of 12, from which we may derive an easy contradiction. ∎

Suppose that V is faithful, irreducible G-module and that $V_C = W_1 \oplus \cdots \oplus W_n$ for C-invariant submodules W_i that are permuted faithfully by the factor group G/C with $n > 1$. Then $C \neq 1$, since otherwise $\mathbf{C}_V(G) \neq 1$. Also W_1 is an irreducible H-module where $H = \mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$, because G is isomorphic as a linear group to an irreducible subgroup of the wreath product $H \operatorname{wr}(G/C)$ (see [MW, Lemma 2.8]). Because $C \neq 1$, also $H \neq 1$ and $|W_1| > 2$.

THEOREM 10. *Suppose that V is a faithful, irreducible G-module and that $V_C = W_1 \oplus W_2 \oplus \cdots \oplus W_p$ for C-invariant submodules W_i that are permuted faithfully and primitively by the factor group $G/C \neq 1$. If G is supersolvable and $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$ is isomorphic to a subgroup of $\Gamma(W_1)$, then G has two distinct semi-regular orbits on V of types ρ and σ with $\rho\sigma > 2$ except possibly when*:

(i) $p = 2$, $|W_1| = 2^2$, *and* $\rho\sigma = 2$;

(ii) $p = 3$, $|W_1| = 2^2$, $G/C \cong S_3$, *and there exist x and y in V in distinct semi-regular orbits of G with y in a type 2 semi-regular orbit and such that $\mathbf{C}_G(x)$ has three distinct orbits in V that lie in regular orbits of $\mathbf{C}_{\mathbf{F}(G)}(x)$; or*

(iii) $p = 2$, $|W_1| = q^2$ *for a Mersenne prime q, and G has a type 3 semi-regular orbit.*

Proof. Because the primitive permutation group $G/C \neq 1$ is supersolvable, p is prime and G/C is isomorphic to a subgroup of $Z_p \circ Z_{p-1}$. Set $H_i = \mathbf{N}_G(W_i)$ and set $C_i = \mathbf{C}_G(W_i)$ and $|W_i| = q^n$ for a prime power $q = |F|$ where F is the underlying field. Because V is an irreducible G-module, also W_1 is an irreducible H_1/C_1-module and $|W_1| \neq 2$ (see comments before the theorem). If H_1/C_1 has a regular orbit on W_1, then G satisfies the principle conclusion of the theorem by Corollary 6. Thus we may assume that H_1/C_1 is not isomorphic to a subgroup of $\Gamma_0(W_1)$ and hence that $n > 1$. If H_1/C_1 has at least two type 2 semi-regular orbits on W_1, then Lemma 5 shows that G has two distinct semi-regular orbits on V of types ρ and σ with $\rho\sigma > 2$. But, by Lemma 7, every non-zero element of W_1 is in a type 2 semi-regular orbit of H_1/C_1 except possibly when $|W_1| = 4$ and $H_1/C_1 \cong \Gamma(2^2) \cong S_3$. Hence, in all cases, we may assume that H_1/C_1 acts transitively on the non-zero vectors of W_1.

We denote by D_i that subgroup of H_i defined by $D_i/C_i = \Gamma_0(W_i) \cap H_i/C_i$. Then D_i/C_i acts on W_i by field multiplications and $C_i = \mathbf{C}_{Di}(x)$ for all non-zero x in W_i. Now, if $|W_i| = q^n$ for a prime power q, then D_i/C_i and H_i/D_i are cyclic with $|D_i/C_i|$ dividing $q^n - 1$ and $|H_i/D_i|$ dividing n. Set $D = \bigcap_{i=1 \text{ to } p} D_i$. Because $\bigcap_{i=1 \text{ to } p} C_i = 1$ and $\bigcap_{i=1 \text{ to } p} H_i = C$, it follows that D and C/D are abelian with exponents dividing $q^n - 1$ and n (respectively). Furthermore, if x_i is a non-zero element of W_i, then $\mathbf{C}_D(x_1 + \cdots + x_p) \subseteq \cap \mathbf{C}_{Di}(x_i) \subseteq \cap C_i = 1$. Unless $q^n = q^2$ for a

Mersenne prime q, then $D_i/C_i = \mathbf{F}(H_i/D_i)$ by Lemma 8 and the Zsigmondy prime theorem. Standard arguments (similar to those of [MW, Proposition 9.5(a)]) then show that $D = \mathbf{F}(C)$ unless $q^n = q^2$ for a Mersenne prime q.

The primitive permutation group G/C has a unique minimal normal subgroup M/C of order p. We claim that C/D is central in M/D. If $\mathbf{F}(G) = \mathbf{F}(C) = D$, then G/D is abelian by Lemma 7. If $\mathbf{F}(G) > \mathbf{F}(C) = D$, then $M/D = C/D \times \mathbf{F}(G)/D$. If $p > n$, then p is the largest prime divisor of $|M/D|$, whence M/D has a normal Sylow-p-subgroup P/D by Proposition 9 and so $M/D = C/D \times P/D$. The claim holds unless $\mathbf{F}(C) > D$ and $p \leq n$. By the last paragraph, this can only happen when q is a Mersenne prime and $n = 2 = p$. But, in this exceptional case, G has a type 3 semi-regular orbit by Proposition 9 and Lemma 2 and thus exception (iii) of this theorem holds. Thus we may assume that C/D is central in M/D.

Now M/C transitively permutes the subgroups $C \cap D_i/D$, while C/D is central in M/D. Thus $C \cap D_1 = \cdots = C \cap D_p$. But $\cap D_i = D$ and so $D = C \cap D_1 = \cdots = C \cap D_p$. Now C/D is isomorphic to a subgroup of H_1/D_1 and is hence cyclic. If $x_i \in W_i$ is non-zero for all i, note that $\mathbf{C}_D(x_i) \subseteq C_i$ and thus $\bigcap_i \mathbf{C}_D(x_i) = 1$. If $u = u_1 + \cdots + u_p$ and $v = v_1 + \cdots + v_p$ with $u_i, v_i \in W_i$, then we claim that $\mathbf{C}_C(u) \cap \mathbf{C}_C(v) = 1$ provided that u_i or v_i is non-zero for each i and that v_j, for some j, is in a regular orbit of $\mathbf{C}_G(u_j)/C_j$. The claim is valid because $\mathbf{C}_D(u) \cap \mathbf{C}_D(v) = 1$ and $\mathbf{C}_C(u) \cap \mathbf{C}_C(v) \subseteq \mathbf{C}_C(u_j) \cap \mathbf{C}_C(v_j) \subseteq C_j \cap C \subseteq D_j \cap C = D$.

If x_i is a non-zero element of W_i, we let r be the number of regular orbits and s be the number of non-zero orbits of $\mathbf{C}_G(x_i)/C_i$ on W_i. These values are independent of choices because H_1/C_1 acts transitively on the non-zero vectors of W_1 and G/C transitively permutes the W_i. Since H_1/C_1 does not have a regular orbit on W_1, $s > r \geq 1$ with the last inequality by Proposition 9.

Fix non-zero $x_i \in W_i$ and set $y = x_1 + \cdots + x_{p-1}$. If $z = z_1 + \cdots + z_p$ with $z_i \in W_i$ with $z_p \neq 0$ and z_{p-1} in a regular orbit of $\mathbf{C}_G(x_{p-1})/C_{p-1}$, then z is in a regular orbit of $\mathbf{C}_C(y)$ by the next to last paragraph. If, in addition, z_{p-1} is not conjugate to z_i for all $i < p-1$, then $\mathbf{C}_G(y) \cap \mathbf{C}_G(z)$ must normalize W_{p-1} and W_p and thus lies in C. With this additional condition, we have that z is in a regular orbit of $\mathbf{C}_G(y)$. Furthermore $a = a_1 + \cdots + a_p$ with $a_i \in W_i$ can only be $\mathbf{C}_C(y)$-conjugate to z if a_i is $\mathbf{C}_G(x_i)/C_i$-conjugate to z_i for $i < p$. Thus we have rs^{p-2} elements of V that lie in regular orbits of $\mathbf{C}_G(y)$ and in distinct orbits of $\mathbf{C}_C(y)$. Since $|\mathbf{C}_G(y)/\mathbf{C}_C(y)|$ divides $p-1$, then $\mathbf{C}_G(y)$ has at least $rs^{p-2}/(p-1) \geq r2^{p-2}/(p-1) \geq r$ distinct regular orbits on V. Thus y is in a type r semi-regular orbit of G. Also, y is in a type 2 regular orbit if $p > 3$. Even when $p = 3$ (and $r = 1$), a slight refinement of the argument shows that $z = z_1 + z_2 + z_3$ is in a regular orbit of $\mathbf{C}_G(y)$ provided $z_3 \neq 0$ and exactly one of z_i for $i < 3$ is in regular orbit of $\mathbf{C}_G(x_i)/C_i$. In this case, we have $2 \cdot 1 \cdot 2/2$ regular orbits of $\mathbf{C}_G(y)$. Summarizing, $\mathbf{C}_G(y)$ has 2 regular orbits on V except possibly when $r = 1$ and $p = 2$. When $r = 1$, then Proposition 9 shows that $|W_1| = 2^2$ and $H_1/C_1 \cong \Gamma(2^2) \cong S_3$.

Next we let $x = x_1 + \cdots + x_p$. If $z = z_1 + \cdots + z_p$ with $z_i \in W_i$ with some z_i in a regular orbit of $\mathbf{C}_G(x_i)/C_i$, then z is in a regular orbit of $\mathbf{C}_C(x)$ by above because every x_i is non-zero. Let $u_i \in W_i$ be in a regular orbit of $\mathbf{C}_G(x_i)/C_i$, so that $u_i \neq 0$ because H_i/C_i has no regular orbit on W_i. Then $u = 0 + \cdots + 0 + x_{p-1} + u_p$ and $v = u_1 + \cdots + u_{p-2} + x_{p-1} + 0$ lie in regular orbits of $\mathbf{C}_G(x)$. If $p > 3$, then u and v cannot even be G-conjugate and $\mathbf{C}_G(x)$ has at least two regular orbits on V. Also $\mathbf{C}_G(x)$ has at least r regular orbits on V, because u and $0 + \cdots + 0 + x_{p-1} + t_p$ can only be $\mathbf{C}_G(x)$-conjugate if t_p and u_p are $\mathbf{C}_G(x_p)/C_p$-conjugate. If G/C is cyclic, then u and $0 + \cdots + 0 + u_p$ are in distinct regular orbits of $\mathbf{C}_G(x)$ on V. Hence $\mathbf{C}_G(x)$ has two regular orbits on V provided $p > 3$, G/C is cyclic or $r > 1$; i.e., x is a type 2 semi-regular orbit unless $p = 3$, $G/C \cong S_3$, $|W_1| = 2^2$, and $H_1/C_1 \cong \Gamma(2^2) \cong S_3$. But do observe in this case that we do have one regular orbit of $\mathbf{C}_G(x)$ on V. Observe, in this exceptional case, that the three elements $0 + x_2 + u_3$, $0 + u_2 + u_3$, and $0 + 0 + u_3$ lie in distinct orbits of $\mathbf{C}_G(x)$ and are in regular orbits of $\mathbf{C}_{\mathbf{F}(G)}(x)$, because $\mathbf{F}(G)/D$ has order three and transitively permutes the W_i and $\mathbf{C}_D(x) = 1$.

Now x and y cannot be G-conjugate because all the x_i are non-zero. Both are in semi-regular orbits of G and at least one lies in a type 2 semi-regular orbit. Indeed both lie in type 2 semi-regular orbits of G or exception (i) or (ii) applies. This completes the proof. ∎

We used the supersolvability of G in critical ways in Theorem 10 (see Example 13). This will be used again in the next two results.

LEMMA 11. *Suppose that V is a faithful G-module and $V = V_1 \oplus \cdots \oplus V_p$ for subspaces that are permuted primitively by G. Suppose that $H_i = \mathbf{N}_G(V_i)$ and that V_1 can be written as a direct sum of $q > 1$ subspaces that are permuted primitively by H_1. If $p > q$ and G is supersolvable, then V can be written as a direct sum of $q > 1$ subspaces that are permuted primitively by G.*

Proof. Let C be the kernel of the permutation action of G on $\{V_1, \dots, V_p\}$. We may write V_1 as a direct sum $W_{11} \oplus \cdots \oplus W_{1q}$ of subspaces primitively and faithfully permuted by H_1/D_1 for a normal subgroup D_1 of H_1. For g, h in the same right coset of H_1 in G, observe that $\{W_{11}, \dots, W_{1q}\}^g = \{W_{11}, \dots, W_{1q}\}^h$. Hence, we may write V_i as a direct sum $W_{i1} \oplus \cdots \oplus W_{iq}$ of subspaces primitively and faithfully permuted by H_i/D_i for a normal subgroup D_i of H_i that is G-conjugate to D_1. Furthermore, if we set $D = \cap D_i \subseteq \cap H_i = C$, then the group G/D permutes the set $\{W_{ij}\}$ of pq subspaces transitively and faithfully.

Since G/C is supersolvable, G/C has a unique minimal normal subgroup M/C of prime order p that transitively permutes the V_i and a cyclic factor group G/M whose order divides $p-1$. Now H_i/D_i has a unique minimal normal subgroup of prime order q and a cyclic factor group whose order divides $q-1$. Since $\cap H_i = C$ and $\cap D_i = D$, it follows from the hypothesis $p > q$ that p must be larger than all prime divisors of $|C/D||G/M|$. By the supersolvability of G, G/D has a normal Sylow-p-subgroup P/C of order p. Now P/C acts on $\{W_{ij}\}$ with q faithful orbits and we may write $V = X_1 \oplus \cdots \oplus X_q$ for P-invariant submodules X_k that are permuted transitively by G/P. If L/P is the kernel of the action of G/P on $\{X_1, \dots, X_q\}$, then G/L even acts primitively on $\{X_1, \dots, X_q\}$ because q is prime. ▮

THEOREM 12. *Suppose that V is a faithful, irreducible G-module for a supersolvable group G. Assume that V is an imprimitive G-module and choose $p > 1$ as small as possible so that V may be written $V = W_1 \oplus \cdots \oplus W_p$ for subspaces W_i that are permuted primitively by G. If W_1 is imprimitive as an $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$-module, then*

(i) $p = 2$ *and G has a type* 3 *semi-regular orbit*; *or*

(ii) *G has two distinct semi-regular orbits of types ρ and σ on V such that $\rho\sigma \geq min\{3, p\}$.*

Proof. Now W_1 is imprimitive as an H-module where $H = \mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$. Choose $q > 1$ as small as possible such that W_1 may be written $W_1 = U_1 \oplus \cdots \oplus U_q$ as a direct sum of q subspaces that are permuted primitively by H/D for a normal subgroup D of H. By Lemma 11 and choice of p, it follows that $q \geq p$.

First suppose that U_1 is imprimitive as a $\mathbf{N}_H(U_1)/\mathbf{C}_H(U_1)$-module. Arguing by induction on $\dim(V)$, we may conclude that:

(a) H has two distinct semi-regular orbits of types r and s on W such that $rs \geq \min\{3, q\}$; or

(b) $q = 2$ and H has a type 3 semi-regular orbit on W.

On the other hand, if U_1 is a primitive $\mathbf{N}_H(U_1)/\mathbf{C}_H(U_1)$-module, then Lemma 7 shows $\mathbf{N}_H(U_1)/\mathbf{C}_H(U_1)$ is isomorphic to a subgroup of $\Gamma(U_1)$. Then we apply Theorem 10 to conclude that H and W satisfy (a) or (b) above or that:

(c) $q = 3$, $|U_1| = 2^2$, H is a $\{2,3\}$-group, and there exist x and y in W_1 in distinct semi-regular orbits of H with y in a type 2 semi-regular orbit and such that $\mathbf{C}_H(x)$ has three distinct orbits in V that lie in regular orbits of regular $\mathbf{C}_{\mathbf{F}(H)}(x)$.

Whether U_1 is primitive or imprimitive, H and W satisfy (a), (b), or (c).

If $q = 2$, then also $p = 2$ by Lemma 11 and choice of p. If H and W satisfy (b), then $p = 2$ and G has a type 3 semi-regular orbit by Lemma 2. Conclusion (i) of the theorem is met here.

If H satisfies (a), then Lemma 5 shows that G has two distinct semi-regular orbits of types ρ and σ on V such that $\rho\sigma \geq \min\{3, q\}$. Conclusion (ii) follows here as $q \geq p$. Similarly, Lemma 5 shows that Conclusion (ii) holds if H satisfies (c) and $p = 2$.

To complete the proof, we may assume that (c) above holds and $p = 3$. Recalling $V = W_1 \oplus W_2 \oplus W_3$, we let $H_i = \mathbf{N}_G(W_i)$, let $C_i = \mathbf{C}_G(W_i)$, and $C = \cap H_i$ so that $H = H_1/C_1$ and $\cap C_i = 1$. Let $F_i/C_i = \mathbf{F}(H_i/C_i)$ and observe that $\cap F_i = \mathbf{F}(C)$. Now G/C primitively and faithfully permutes $\{W_1, W_2, W_3\}$ and is isomorphic to a transitive subgroup of S_3. Because G is a supersolvable $\{2,3\}$-group, G has a normal Sylow-3-subgroup. But $\operatorname{char}(V) = 2$ and V is irreducible, so that $\mathbf{O}_2(G) = 1$ and $\mathbf{F}(G)$ is the Sylow-3-subgroup of G. Now $\mathbf{F}(G)/\mathbf{F}(C)$ is isomorphic to the Sylow-3-subgroup M/C of G/C and transitively permutes $\{W_1, W_2, W_3\}$. Also $C/\mathbf{F}(C)$ is the 2-group and so $M/\mathbf{F}(C) = C/\mathbf{F}(C) \times \mathbf{F}(G)/\mathbf{F}(C)$. Since $\mathbf{F}(G)/\mathbf{F}(C)$ transitively permutes the groups $F_i \cap C/\mathbf{F}(C)$ and centralizes $C/\mathbf{F}(C)$, it follows that $F_1 \cap C = F_2 \cap C = F_3 \cap C = \mathbf{F}(C)$.

By (c) above, we may choose $x_i, y_i \in W_i$ in distinct semi-regular orbits in W_i of H_i/C_i of types 1 and 2 (respectively) and such that $\mathbf{C}_G(x_i)/C_i \subseteq H_i/C_i$ has three distinct orbits on W_i that lie in regular orbits of $\mathbf{C}_G(x_i) \cap F_i/C_i$. We may assume that no x_i and y_k are G-conjugate. Suppose that s_i, $t_i \in W_i$ lie in distinct regular orbits of $\mathbf{C}_G(y_i)/C_i$ for each i. Also choose $a_i, b_i, c_i \in W_i$ in distinct orbits of $\mathbf{C}_G(x_i) \subseteq H_i$, each of which is in a regular orbit $\mathbf{C}_G(x_i) \cap F_i/C_i$. For $i \neq j$, we may assume that a_i and b_j are not conjugate via $\mathbf{C}_G(x_i + x_j)$, etc. Then $\mathbf{C}_G(y_1 + x_2 + x_3) \cap \mathbf{C}_G(s_1 + b_2 + c_3) \subseteq \cap H_i = C$. Now $\mathbf{C}_C(y_1 + x_2 + x_3) \cap \mathbf{C}_C(s_1 + b_2 + c_3) \subseteq \mathbf{C}_C(y_1) \cap \mathbf{C}_C(s_1) \subseteq C_1 \cap C \subseteq F_1 \cap C = \mathbf{F}(C)$. Because $\mathbf{F}(C) \subseteq F_i$, it follows that $\mathbf{C}_{\mathbf{F}(C)}(y_1 + x_2 + x_3) \cap \mathbf{C}_{\mathbf{F}(C)}(s_1 + b_2 + c_3) \subseteq \cap C_i = 1$. Hence $\mathbf{C}_G(y_1 + x_2 + x_3) \cap \mathbf{C}_G(s_1 + b_2 + c_3) = 1$ and so $s_1 + b_2 + c_3$ is in a regular orbit of $\mathbf{C}_G(y_1 + x_2 + x_3)$. Likewise, $t_1 + b_2 + c_3$ and $s_1 + a_2 + c_3$ are in regular orbits of $\mathbf{C}_G(y_1 + x_2 + x_3)$. Thus $y_1 + x_2 + x_3$ lies in a type 3 semi-regular orbit of G (see Proposition 1). Now similar arguments show that $s_1 + t_2 + a_3$, $s_1 + t_2 + b_3$, and $s_1 + t_2 + c_3$ lie in regular orbits of $\mathbf{C}_G(y_1 + y_2 + x_3)$ and these lie in distinct orbits of $\mathbf{C}_G(y_1 + y_2 + x_3)$ by Proposition 1. Both $y_1 + y_2 + x_3$ and $y_1 + x_2 + x_3$ lie in a type 3 semi-regular orbits of G. Because $y_1 + x_2 + x_3$ and $y_1 + y_2 + x_3$ are not G-conjugate, conclusion (ii) of this theorem is met. ∎

Proof of Theorem A. Arguing by induction on $\dim(V)$, we may assume that V is irreducible. If V is a primitive G-module, then $G \subseteq \Gamma(V)$ by Lemma 7, whence Proposition 9 gives the existence of a semi-regular orbit.

Otherwise we may write $V = W_1 \oplus \cdots \oplus W_p$ for subspaces W_i that are permuted primitively by G and do so with p as small as possible. If W_1 is imprimitive as a $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$-module, then Theorem 12 gives the existence of a semi-regular orbit. If W_1 is primitive as a $\mathbf{N}_G(W_1)/\mathbf{C}_G(W_1)$-module, then Theorem 10 gives the existence of a semi-regular orbit. ∎

We next give an example where Theorem A fails for a group G that is the wreath product $H \operatorname{wr} S$ of supersolvable groups acting on $V = W^G$, but where the conclusion of Theorem A is valid for the action of H on W. While the action here is not coprime, Isaacs' question of an orbit as large as $|G|^{1/2}$ also fails here.

EXAMPLE 13. Let $H = S_3 = S$, so that H acts irreducibly on a vector space W of order 2^2 and has exactly one semi-regular orbit. The wreath product $G = H \operatorname{wr} S$ of order $2^4 3^4$ acts irreducibly on a vector space V of order 2^6. The G-orbits in V have size 1, 9, 27, and 27 and so no G-orbit has size as large as $|G|^{1/2}$. Furthermore, for each $v \in V$, $|\mathbf{C}_G(v)|$ has order at least $2^4 3$. Because $|\mathbf{C}_G(v)|$ is larger than any G-orbit on V, indeed $\mathbf{C}_G(v)$ has no regular orbits on V. Thus $\mathbf{C}_G(v) \cap \mathbf{C}_G(w) \neq 1$ for all $v, w \in V$.

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